

On Rivest-Vuillemin Conjecture for Fourteen Variables

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Abstract

A boolean function $f(x_1, \dots, x_n)$ is *weakly symmetric* if it is invariant under a transitive permutation group on its variables. A boolean function $f(x_1, \dots, x_n)$ is *elusive* if we have to check all x_1, \dots, x_n to determine the output of $f(x_1, \dots, x_n)$ in the worst-case. It is conjectured that every nontrivial monotone weakly symmetric boolean function is elusive, which has been open for a long time. In this paper, we show that this conjecture is true for $n = 14$.

Keywords: Boolean function, Decision tree complexity, Elusiveness

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1. Introduction

A boolean function $f(x_1, \dots, x_n)$ can be computed by a binary tree where each non-leaf node is labeled by a variable and the leaves are labeled by 0 or 1. For each non-leaf node, the two edges linking to its left and right child are
 5 labeled by 1 and 0, respectively. For any path from root to leaf, every variable appears at most once. An input \mathbf{x} of f is a subset of $\{x_1, \dots, x_n\}$ where $x_i = 1$ if and only if $x_i \in \mathbf{x}$. For each input \mathbf{x} of f , its value can be computed according to the decision tree of $f(x_1, \dots, x_n)$. That is, starting from the root, if the label of root is in \mathbf{x} , we go to its left child; otherwise we go to its right child. The
 10 above process is repeated until a leaf is reached and the function value of x is given by the leaf's label. For each input \mathbf{x} , the computation time depends on the length of the corresponding root-leaf path, i.e., the number of checked

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variables. The depth of a decision tree is the maximum length among all root-leaf paths. One can see that for a certain boolean function f , there can be
15 more than one decision trees. We denote by $D(f)$ the minimum depth among all its decision trees. A boolean function f of n variables is called elusive if $D(f) = n$. In other words, if f is elusive, then for each of its decision trees, there exists an input \mathbf{x} such that deciding $f(\mathbf{x})$ requires checking all the variables. A boolean function f is monotone non-increasing if $f(\mathbf{x}) = 1$ implies $f(\mathbf{x}') = 1$ for
20 each $\mathbf{x}' \subseteq \mathbf{x}$, and, similarly, f is monotone non-decreasing if $f(\mathbf{x}) = 0$ implies $f(\mathbf{x}') = 0$ for each $\mathbf{x}' \subseteq \mathbf{x}$. For a permutation σ on $\{1, \dots, n\}$ and an input $\mathbf{x} = \{x_{a_1}, \dots, x_{a_m}\}$, let $\sigma(\mathbf{x}) = \{x_{\sigma(a_1)}, \dots, x_{\sigma(a_m)}\}$. A boolean function f is σ -invariant if $f(\mathbf{x}) = f(\sigma(\mathbf{x}))$ for every \mathbf{x} . For a group G of permutations, f is called G -invariant if f is σ -invariant for every $\sigma \in G$. The symmetry of f
25 is characterized by its invariant group. An G -invariant boolean function f is weakly symmetric if G is transitive on $\{1, \dots, n\}$.

Rivest-Vuillemin conjecture: every nontrivial monotone weakly symmetric boolean function is elusive.

In [1], [2] and [3], it has been shown that it is also true when $n = 6, 10$ and
30 12. Therefore, the Rivest-Vuillemin conjecture is true for n less than 14. In this paper, we consider $n = 14$.

2. Preliminaries

Let \bar{f} be the opposite function of f , i.e., $\bar{f}(\mathbf{x}) = 0$ iff $f(\mathbf{x}) = 1$. It can be easily seen that $D(\bar{f}) = D(f)$. Therefore, to prove the Rivest-Vuillemin conjecture, it suffices to consider monotone non-increasing boolean functions. Each monotone non-increasing boolean function $f(x_1, \dots, x_n)$ can be equivalently represented as an abstract simplicial complex Δ_f on n vertices defined by $\Delta_f = \{x | x \subseteq \{x_1, \dots, x_n\} \text{ and } f(x) = 1\}$. The faces in Δ_f correspond to the true inputs of f . For an abstract simplicial complex Δ , the Euler characteristic $\chi(\Delta)$ is defined as

$$\chi(\Delta) = \sum_{i=1}^n (-1)^{i+1} r(\Delta, i), \quad (1)$$

where $r(G, i) = |\{x \in \Delta \mid |x| = i\}|$. Note that if f is G -invariant then G is a group of automorphisms on Δ_f . Kahn *et al.* [4] first observe that the evasiveness
 35 of a monotone boolean function f is related to the topological property of Δ_f .

Theorem 1. ([4]) *If a monotone boolean function f is not evasive, then Δ_f is collapsible and therefore contractible and Z_p -acyclic.*

For two primes p and q , we denote by Ψ_p^q the class of the finite group G with a normal subgroup $P \triangleleft H \triangleleft G$, such that P is of p -power order, the quotient
 40 group G/H is of q -power order, and the quotient group H/P is cyclic; denote by Ψ_p the class of the finite group G with a normal p -subgroup $P \triangleleft G$ such that the quotient group G/P is cyclic. The following fixed-point theory is attributed to Oliver,

Theorem 2. ([5]) *For a collapsible abstract complex Δ with a group G of automorphisms on Δ , if G is cyclic or $G \in \Psi_p$ for some prime p , then $\chi(\Delta^G) = 1$; if $G \in \Psi_p^q$, then $\chi(\Delta^G) \equiv 1 \pmod{q}$, where*

$$\Delta^G = \{\{H_1, \dots, H_k\} \mid H_1, \dots, H_k \text{ are orbits of } G \text{ and } H_1 \cup \dots \cup H_k \in \Delta\} \cup \{\emptyset\}.$$

We call the groups in Ψ_p^q and Ψ_p as Oliver groups. The following result
 45 directly follows from Theorems 1 and 2,

Theorem 3. *For a monotone non-increasing G -invariant boolean function f , if G is transitive, and, G is cyclic or G is an Oliver group, then f is elusive or trivial.*

Proof. If f is not evasive, then Δ_f is collapsible and thus, by Theorem 2, Δ_f^G is
 50 non-empty. Since G is transitive, that Δ_f^G is non-empty implies that the only orbit of G is in Δ_f^G , which means $\{x_1, \dots, x_n\} \in \Delta_f$ and therefore f is trivial. \square

3. Main result

In this paper, we show the following result.

Table 1: Minimal transitive permutation groups of degree 14

Group	index	Generators	Order
G_1	(1)	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14)	14
G_2	(2)	(1, 3, 5, 7, 9, 11, 13)(2, 4, 6, 8, 10, 12, 14), (1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)(13, 14)	14
G_3	(6)	(3, 10)(5, 12)(6, 13)(7, 14), (1, 3, 4, 7, 9, 11, 13)(2, 4, 6, 8, 10, 12, 14)	56
G_4	(12)	(1,3,5,7,9,11,13), (1,2)(3,14,13,4)(5,12,11,6)(7,10,9,8), (1, 6, 13, 8)(2, 9, 12, 5)(3, 4, 11, 10)(7, 14)	169
G_5	(30)	(2, 4, 6, 8, 10, 12, 14), (2, 4, 8)(6, 12, 10), (1, 8)(2, 5)(3, 4)(6, 13)(7, 14)(9, 12)(10, 11)	1092
G_6	(10)	(1, 5, 11, 10)(2, 9)(3, 8, 12, 4)(6, 14, 13, 7), (1, 9, 5, 14)(2, 12, 7, 8)(3, 4, 10, 11)(6, 13)	168

*The second column shows the index of each group in the GAP system.

Theorem 4. *Every nontrivial monotone non-increasing weakly symmetric boolean function of 14 variables is elusive.*

According to [6], there are totally 63 transitive groups of degree 14 up to permutation isomorphism, where there are 6 minimal transitive groups shown in Table 1. These groups can be found in GAP system ([7]). Let G_i , $1 \leq i \leq 6$, be the i^{th} minimal transitive group. Therefore, any weakly symmetric boolean function with 14 variables must be invariant under at least one of the groups of G_i . Thus, to prove Theorem 4, it suffices to show that every nontrivial G_i -invariant monotone non-increasing boolean function is elusive. In the following, we will show that the first four groups are either cyclic or Oliver groups, which can be handled by Theorem 4, while the last two groups are neither cyclic nor Oliver groups for which we propose new techniques. In the rest of this section, G_1 , G_2 , G_3 and G_4 will be considered in Sec. 3.1, and, G_5 and G_6 will be discussed in Sec. 3.2, respectively.

3.1. G_1, G_2, G_3 and G_4

Lemma 1. *Every non-trivial monotone non-increasing G_1 -invariant boolean function f is elusive.*

Proof. Since G_1 is cyclic, the lemma directly follows from Theorem 3. \square

Lemma 2. *Every non-trivial monotone non-increasing G_2 -invariant boolean function f is elusive.*

Proof. Let

$$a = (1, 3, 5, 7, 9, 11, 13)(2, 4, 6, 8, 10, 12, 14)$$

and

$$b = (1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)(13, 14).$$

$G_2 = \langle a, b \rangle$. Let $P = \langle a \rangle$ be the subgroup of G_2 generated by a . Since $bab = a^6$, P has an index of 2. Therefore, P is a normal 7-subgroup and G/P is a cyclic group. Thus, $G_2 \in \Psi_7$. By Theorem 4, every G_2 -invariant monotone boolean function f is elusive. \square

Lemma 3. *Every non-trivial monotone non-increasing G_3 -invariant boolean function f is elusive.*

Let

$$a = (2, 9)(4, 11)(5, 12)(6, 13), \quad b = (1, 8)(2, 9)(5, 12)(7, 14)$$

and

$$c = (3, 10)(5, 12)(6, 13)(7, 14).$$

One can check that group $\langle a, b, c \rangle$ is a normal 2-subgroup of G_3 of index 7. Therefore $G_3 \in \Psi_2$ and by Theorem 4, every nontrivial G_3 -invariant monotone boolean function is elusive.

Lemma 4. *Every non-trivial G_4 -invariant monotone non-increasing boolean function f is elusive.*

Let

$$a = (3, 13)(4, 14)(5, 11)(6, 12)(7, 9)(8, 10), \quad b = (1, 3, 5, 7, 9, 11, 13)$$

and

$$c = (2, 4, 6, 8, 10, 12, 14).$$

Group $H = \langle a, c, b \rangle$ is a normal subgroup of G_4 . Because $|H| = 98$ and $|G_5| = 196$, G/H is of order 2. Let

$$d = (1, 3, 5, 7, 9, 11, 13)$$

and

$$e = (2, 4, 6, 8, 10, 12, 14).$$

85 Group $P = \langle d, e \rangle$ is a normal subgroup of H . Because $|H| = 98$ and $|P| = 49$, P is of 7-power and H/P is cyclic. Therefore, $G_5 \in \Psi_7^2$, and thus, by Theorem 4, proved.

3.2. G_5 and G_6

In this section we consider G_5 and G_6 . Note that G_5 and G_6 are not cyclic
90 and furthermore they are not solvable. Thus, the existing technique can not be applied to prove the evasiveness of an G_5 or G_6 -invariant monotone boolean function. In the following, we proceed in another approach.

The following result is well-known and intuitive.

Lemma 5. *For a G -invariant boolean function $f(x_1, \dots, x_n)$ where G is transi-*
95 *tive, if, for some $x_a \in \{x_1, \dots, x_n\}$, $f_{x_a=1}$ is elusive, then f is elusive.*

Proof. Let T be an arbitrary decision tree of f and σ a permutation in G . By relabeling the variable x_i in T by $x_{\sigma(i)}$, we obtain another decision tree denoted by $\sigma(T)$. Since the label of leaves remain unchanged, T and $\sigma(T)$ have the same length. Suppose the root of T is x_b . Because G is transitive, there is a
100 permutation $\bar{\sigma}$ in G such that $\bar{\sigma}(b) = a$. Therefore, $\bar{\sigma}(T)$ is a decision tree of f with root x_a . Let $\bar{\sigma}(T)_a^1$ be the left subtree of $\bar{\sigma}(T)$ on the root x_a . Because

$f_{x_a=1}$ is elusive and $\bar{\sigma}(T)_a^1$ is a decision tree of $f_{x_a=1}$, the depth of $\bar{\sigma}(T)_a^1$ is $n-1$, which implies $\bar{\sigma}(T)$ is of depth n and so is T . Since T is arbitrarily selected, every the decision tree of f has a depth of n . \square

105 One can check that if $f_{x_a=1}$ is elusive for some x_a , then $f_{x_i=1}$ is elusive for every x_i . Although it is not easy to directly prove the elusiveness of an G_5 -invariant function f , we are able to show that $f_{x_a=1}$ is elusive for some x_a .

Lemma 6. *If a nontrivial monotone Boolean function f of $p+1$ variables is invariant under a transitive group G where p is a prime and $|G| = p * k$ where*
110 *k is an integer and p does not divide k , then f is elusive.*

Proof. By the Sylow p -subgroup theory, G has cyclic subgroup H such that $|H| = p$ and H is the stabilizer of some x_a . Therefore, $f_{x_a=1}$ is invariant under H . Since H is cyclic and transitive on $\{x_1, \dots, x_n\} \setminus \{x_a\}$, $f_{x_a=1}$ is elusive according to Theorem 4. Combining Lemma 5, f is elusive. \square

115 Since $|G_5| = 13 * 3 * 7 * 2^2$, the following directly follows.

Corollary 1. *Every non-trivial G_5 -invariant monotone boolean function is elusive.*

Now let us consider G_6 .

First we consider the subgroups of G_6 . Note that if f is invariant under
120 G_6 then it is also invariant under any subgroup of G_6 . Therefore, if f is not elusive, then by Theorem 1 Δ_f is collapsible and thus for any subgroup $G' < G$, if G' is cyclic or $G' \in \Psi_p$, $\chi(\Delta^{G'}) = 1$; if $G' \in \Psi_p^q$, $\chi(\Delta^{G'}) \equiv 1 \pmod{q}$. Now let us consider the 11 subgroups of G_6 listed in Table A.2, A.3 and A.4 in Appendix A, denoted by G_6^1, \dots, G_6^{11} . By the above analysis, if an G_6 -invariant
125 monotone boolean function f is not elusive, then $\chi(\Delta_f^{G_6^i}) = 1$ for $1 \leq i \leq 10$ and $\chi(\Delta_f^{G_6^{11}}) \equiv 1 \pmod{2}$. Note that when G' is identity, $\chi(\Delta_f^{G'}) = 1$ implies $\chi(\Delta_f) = 1$.

Second, we consider f restricted on one of its variables. For every $x_a \in \{x_1, \dots, x_n\}$, let $\text{Link}(\Delta, x_a)$ and $\text{Deletion}(\Delta, x_a)$ be the subcomplexes of Δ ,

which are defined as

$$\text{Link}(\Delta, x_a) = \{x - \{x_a\} \mid x_a \in x, x \in \Delta\},$$

and

$$\text{Deletion}(\Delta, x_a) = \{x \mid x_a \notin x, x \in \Delta\}.$$

It can be easily checked that $\text{Link}(\Delta_f, x_a) = \Delta_{f_{x_a=1}}$. Thus, if f is not elusive, then $f_{x_a=1}$ is not elusive and therefore $\Delta_{f_{x_a=1}}$ is collapsible, which implies $\chi(\Delta_{f_{x_a=1}}) = 1$. Due to the weakly symmetry, once $r(\Delta_f, k)$ is known to us, the following relationship allows us to compute $r(\Delta_{f_{x_a=1}}, k)$ efficiently,

$$n \cdot r(\Delta_{f_{x_a=1}}, k) = k \cdot r(\Delta_f, k).$$

By the above analysis, if f is G_6 -invariant but not elusive, the followings must be satisfied:

- 130 • $\chi(\Delta_f^{G_6^i}) = 1$ for $1 \leq i \leq 10$ and $\chi(\Delta_f^{G_6^{11}}) \equiv 1 \pmod{2}$;
- $\chi(\Delta_{f_{x_1=1}}) = 1$.

Our goal is to verify that such an f does not exist, i.e.,

Theorem 5. *There is no monotone non-increasing G_6 -invariant boolean function f such that $\chi(\Delta_f^{G_6^i}) = 1$ for $1 \leq i \leq 10$, $\chi(\Delta_f^{G_6^{11}}) \equiv 1 \pmod{2}$, and*
 135 $\chi(\Delta_{f_{x_1=1}}) = 1$.

To this end, let us consider the pattern of the orbits generated by G_6 . We call a k -subsets of $\{x_1, \dots, x_{14}\}$ as a k -tuple. Let T_k be the set of all k -tuples. The orbits on the k -tuples generated by G_6 are called k -orbits. A k -orbit is a subset of T_k . For example, G_6 forms two 2-orbits on the 2-tuples where one orbit has 84 elements and another one has 7 elements. For a k_1 -orbit O_1 and a k_2 -orbit O_2 where $k_1 < k_2$, if there exists two tuples t_1 and t_2 such that $t_1 \in O_1$, $t_2 \in O_2$ and $t_1 \subseteq t_2$, we say O_1 is smaller than O_2 or equivalently O_2 is larger than O_1 , denoted by $O_1 \leq O_2$. Let $\text{Upper}(O)$ be the set of orbits which are larger than orbit O , i.e, $\text{Upper}(O) = \{O' \mid O \leq O'\}$, and similarly, let

145 Lower(O) = $\{O' | O' \leq O\}$. Because f is invariant under G_6 , the tuples in the same k-orbit must have the same function value. We say a k-orbit is a T-orbit (resp. F-orbit) if the tuples in it result true (resp. false) function value. Due to the monotonicity, if an orbit O is a T-orbit, then the orbits in Lower(O) must be T-orbits; if an orbit O is an F-orbit, then the orbits in Upper(O) must be F-orbits. The relationship between the orbits under G_6 are shown in Fig. 1, 150 where if there is an edge between two orbits, then one is larger than the other. Since G_6 is given explicitly, the relationship of orbits can be easily computed by program. We number the orbits consistently and let $O_{i,j}$ be the j -th i -orbit, $1 \leq i \leq 14$ and $j \geq 0$.

155 As shown in Fig. 1, there are totally 158 orbits under G_6 , which means there are 2^{158} boolean functions invariant under G_6 . Thus, it is impracticable to directly check all those functions. In the following, we will show that it suffices to consider a small number of functions, due to a case by case analysis. Initially, all the orbits are called free orbits. In order to satisfy the conditions in Theorem 160 5, some orbits have to be determined as T-orbits or F-orbits. For example, because $\chi(\Delta_f^{G_6^{11}}) \equiv 1 \pmod{2}$ and G_6^{11} forms two orbits H_{11}^1 and H_{11}^2 on 1-tuples, $\Delta_f^{G_6^{11}}$ is either $\{\{H_{11}^1\}, \emptyset\}$, $\{\{H_{11}^2\}, \emptyset\}$ or $\{\{H_{11}^1\}, \{H_{11}^2\}, \{H_{11}^1, H_{11}^2\}, \emptyset\}$. However, f is nontrivial which means $\{H_{11}^1, H_{11}^2\} \notin \Delta_f$. Therefore only $\{\{H_{11}^1\}, \emptyset\}$ and $\{\{H_{11}^2\}, \emptyset\}$ are possible. If $\{\{H_{11}^1\}, \emptyset\}$ is the case, then according to Table A.4, the 6-tuple $H_{11}^1 = \{4, 5, 11, 7, 14, 12\}$ must be a true input and $H_{11}^2 = \{1, 9, 3, 10, 6, 8, 2, 12\}$ must be false input. Because H_{11}^1 and H_{11}^2 belongs to orbits $O_{6.24}$ and $O_{8.24}$, respectively, orbits in Lower($O_{6.24}$)¹ should be T-orbits and the orbits in Upper($O_{8.24}$) should be F-tuples. Therefore, by checking the conditions in Theorem 5 we can keep determining the type of the orbits. After 170 checking $\chi(\Delta_f^{G_6^1})$, there will be no free orbits, which implies the function is completely determined. Finally, we can check $\chi(\Delta_{f_{x_1=1}})$ to see whether there is an G_6 -invariant function satisfying all the conditions in Theorem 5. Specifically,

¹The index of the orbits does not matter as long as long it is consistent. Here we use the index generated by our program for illustration.

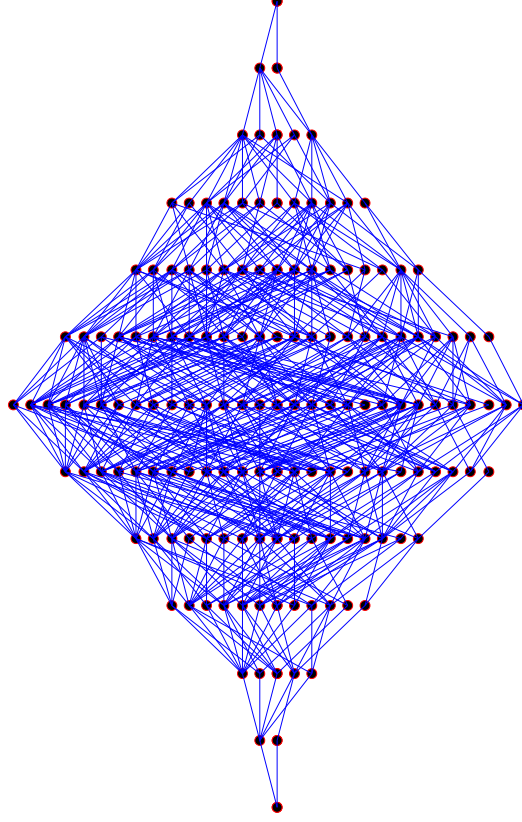


Figure 1: **Orbits pattern of G_6**

we will first check the $\chi(\Delta_f^{G_6^i})$ where G_6^i has the fewest orbits. The checking framework is shown in Algorithm 1. The whole process is done by a Java-based
175 programming combining with the GAP system. It turns out that no type setting of the orbits can satisfy all the conditions. In the appendix, we provide an example to show one branch of the computing.

4. Discussion

There has been other works that manage to verify the elusiveness of a boolean
180 function by programming. For example, in [8], the authors have checked the

Algorithm 1 check(T-orbits, F-orbits, index)

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1: if (index  $\leq 11$ ) then
2:   for each feasible case such that  $\chi(\Delta_f^{G^{index}}) = 1$  do
3:     update T-orbits, F-orbits;
4:     check(T-orbits, F-orbits, index+1);
5: else
6:   if (index == 12) then
7:     compute  $\chi(\Delta_{f_{x_1=1}}^{G_6})$  according to  $T$ -orbits,  $F$ -orbits.
8:     if  $\chi(\Delta_{f_{x_1=1}}^{G_6}) == 1$  then
9:       return a feasible boolean function found;

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evasiveness of a G -variant boolean function for some G by enumerating the complexes and checking the Z_p -acyclic. However, given a group G , checking all the G -invariant boolean functions in brute force is extremely time consuming and the method proposed in [8] cannot deal with the case for $14 \leq n$. The
185 checking framework proposed in this paper in more efficient and fundamentally it reveals how the weakly symmetry forces the complex to be a simplex.

The initial conjecture made by Rivest and Vuillemin [9] is that every weakly symmetric boolean function f with $f(\emptyset) \neq f(\{x_1, \dots, x_n\})$ is elusive, which is negated by Illies by a counterexample [4]. Aigner [10] further modify the con-
190 jecture into its current version by adding the condition of monotonicity. Due to the monotonicity, a boolean function f is equivalent to an abstract simplicial complex Δ_f . The critical observation by Kahn et al. [4] shows if f is non-elusive then Δ_f must be collapsible and therefore contractible, which enables us to apply the fixed-point theory. For a contractible abstract simplicial complex with a
195 automorphism group G , Oliver [5] shows that under certain circumstance (i.e., Oliver group) there exists a face which is fixed by G . Therefore, if the invariant group is an Oliver transitive group, Δ_f must be a simplex, which means f is trivial. When G is not a Oliver group, we may apply the fixed-point theory to its subgroups, as shown in this paper. Given the invariant group, we have although
200 large but limited number of boolean functions. While applying the fixed-point

theory to the subgroups, we are able to eliminate the complexes that are not collapsible. Kahn et al. [4] propose a conjecture that a non-empty collapsible weakly symmetric complex must be a simplex. The truth of this conjecture yields the truth of Revest-Vuillemin conjecture².

205 Finally, we remark a stronger condition. Note that the Link and Deletion of a non-evasive weakly symmetric complex must be non-evasive. Thus, the following conjecture implies the Revest-Vuillemin conjecture:

Conjecture 1. *For a non-empty weakly symmetric complex Δ , if $\text{Link}(\Delta, x)$ and $\text{Deletion}(\Delta, x)$ are all collapsible, then Δ is a simplex.*

210 The condition in the above statement is stronger and it has a clear meaning that the complex is not only collapsible but also be able to collapse to a point along a certain sequence of collapses.

²As mentioned in [4], Oliver has provided a plausibility argument for the falsity of this conjecture, in personal communication.

Appendix A. Subgroups of G_6

Table A.2: Subgroups of G_6

Group	index	Generators and orbits	type
G_6^1	(1)	identity	
G_6^2	(2)	Generators: $(2,4)(3,10)(5,6)(7,14)(9,11)(12,13)$ orbits: $H_2^1: \{1\} \in O_{1.0}; \quad H_2^2: \{2, 4\} \in O_{2.0};$ $H_2^3: \{3, 10\} \in O_{2.1}; \quad H_2^4: \{5, 6\} \in O_{2.0};$ $H_2^5: \{7, 14\} \in O_{2.1}; \quad H_2^6: \{8\} \in O_{1.0};$ $H_2^7: \{9, 11\} \in O_{2.0}; \quad H_2^8: \{12, 13\} \in O_{2.0}$	cyclic
G_6^3	(23)	Generators: $(2,3,6)(4,7,12)(5,11,14)(9,10,13)$ orbits: $H_3^1: \{1\} \in O_{1.0}; \quad H_3^2: \{2, 3, 6\} \in O_{3.2};$ $H_3^3: \{4, 7, 12\} \in O_{3.1}; \quad H_3^4: \{5, 11, 14\} \in O_{3.4};$ $H_3^5: \{8\} \in O_{1.0}; \quad H_3^6: \{9, 10, 13\} \in O_{3.2}$	cyclic
G_6^4	(51)	Generators: $(1,8)(2,13)(3,10)(4,12)(5,11)(6,9),$ $(1,8)(2,12)(4,13)(5,9)(6,11)(7,14)$ orbits: $H_4^1: \{1, 8\} \in O_{2.1}, \quad H_4^2: \{2, 13, 12, 4\} \in O_{4.10},$ $H_4^3: \{3, 10\} \in O_{2.1}, \quad H_4^4: \{5, 11, 9, 6\} \in O_{4.0},$ $H_4^5: \{7, 14\} \in O_{2.1}$	Ψ_2
G_6^5	(58)	Generators: $(2,11)(3,7)(4,9)(5,12)(6,13)(10,14),$ $(2,4)(3,10)(5,6)(7,14)(9,11)(12,13)$ orbits: $H_5^1: \{1\} \in O_{1.0}, \quad H_5^2: \{2, 4, 11, 9\} \in O_{(4.11)},$ $H_5^3: \{3, 10, 7, 14\} \in O_{4.11},$ $H_5^4: \{5, 6, 12, 13\} \in O_{4.11}, \quad H_5^5: \{8\} \in O_{1.0}$	Ψ_2

Table A.3: Subgroups of G_6 (cont.)

Group	index	Generators and orbits on 1-tuples	type
G_6^6	(65)	Generators: $(1,8)(2,5,4,6)(3,7,10,14)(9,12,11,13)$ orbits: $H_6^1: \{1, 8\} \in O_{2.1}$; $H_6^2: \{2, 5, 4, 6\} \in O_{4.7}$, $H_6^3: \{3, 7, 10, 14\} \in O_{4.11}$, $H_6^4: \{9, 12, 11, 13\} \in O_{4.7}$	cyclic
G_6^7	(86)	Generators: $(2,3,6)(4,7,12)(5,11,14)(9,10,13)$ $(1,8)(2,13)(3,10)(4,12)(5,11)(6,9)$ orbits: $H_7^1: \{1, 8\} \in O_{2.1}$, $H_7^2: \{2, 3, 13, 6, 10, 9\} \in O_{6.23}$, $H_7^3: \{4, 7, 12\} \in O_{3.1}$, $H_7^4: \{5, 11, 14\} \in O_{3.4}$	Ψ_2
G_6^8	(142)	Generators: $(1,14)(2,9)(4,6)(5,12)(7,8)(11,13)$ $(1,8)(2,10)(3,9)(4,7)(6,13)(11,14)$ orbits: $H_8^1: \{1, 14, 8, 11, 7, 13, 4, 6\} \in O_{8,24}$, $H_8^2: \{2, 9, 10, 3\} \in O_{4.11}$, $H_8^3: \{5, 12\} \in O_{2.1}$	Ψ_2
G_6^9	(149)	Generators: $(1,11,3)(2,6,14)(4,10,8)(7,9,13)$ $(1,3)(2,9)(4,5)(6,13)(8,10)(11,12)$ orbits: $H_9^1: \{1, 11, 3, 12\} \in O_{4.10}$, $H_9^2: \{2, 6, 9, 14, 13, 7\} \in O_{6.24}$, $H_9^3: \{4, 10, 5, 8\} \in O_{4.10}$	Ψ_2

Table A.4: Subgroups of G_6 (cont.)

Group	index	Generators and orbits on 1-tuples	type
G_6^{10}	(157)	Generators: $(1,6,4)(2,12,3)(5,10,9)(8,13,11),$ $(1,12,6)(3,7,4)(5,13,8)(10,14,11)$ orbits: $H_{10}^1: \{1, 6, 12, 4, 3, 2, 7\} \in O_{7.20},$ $H_{10}^2: \{5, 10, 13, 9, 14, 11, 8\} \in O_{7.27}$	Ψ_7
G_6^{11}	(165)	Generators: $(1,9,10)(2,3,8)(4,5,7)(11,12,14),$ $(1,3,9,6)(2,13,8,10)(4,11)(5,14,12,7)$ orbits: $H_{11}^1: \{4, 5, 11, 7, 14, 12\} \in O_{6.24},$ $H_{11}^2: \{1, 9, 3, 10, 6, 8, 2, 13\} \in O_{8.24}$	Ψ_2^2

Appendix B. A case study for Algorithm 1

215 **Step 1:** As discussed in Sec. 3.2, in order to meet that $\chi(\Delta_f^{G_6^{11}}) \equiv 1 \pmod{2}$, there are two cases to consider. Suppose $\chi(\Delta_f^{G_6^{11}}) = \{\{H_{11}^1\}, \emptyset\}$ is selected. Let Θ_T and Θ_F be the set of the T-orbits and F-orbits that be currently determined. Thus, $\Theta_T = \text{Lower}(O_{6.24})$ and $\Theta_F = \text{Upper}(O_{8.24})$. According to the relationship in Fig. 1, currently,

$$\Theta_T = \{O_{1.0}, O_{2.0}, O_{2.1}, O_{3.1}, O_{1.3}, O_{1.4}, O_{4.9}, O_{4.11}, O_{5.16}, O_{6.24}\}.$$

$$\Theta_F = \{O_{8.24}, O_{9.16}, O_{10.9}, O_{10.11}, O_{11.1}, O_{11.3}, O_{11.4}, O_{12.0}, O_{12.1}, O_{13.0}, O_{14.0}\}.$$

220 **Step 2:** Now we consider $\Delta_f^{G_6^{10}}$. Note that $H_{10}^1 \in O_{7.20}$ and $H_{10}^2 \in O_{7.27}$. Because neither of $O_{7.20}$ or $O_{7.27}$ is in Θ_T or Θ_F , we have two cases to consider. One is $\chi(\Delta_f^{G_6^{10}}) = \{\{H_{10}^1\}, \emptyset\}$ and the other is $\chi(\Delta_f^{G_6^{10}}) = \{\{H_{10}^2\}, \emptyset\}$. Suppose $\chi(\Delta_f^{G_6^{10}}) = \{\{H_{10}^1\}, \emptyset\}$ is true. Now more orbits can be determined as T- or F-orbits. In particular, $\Theta_T = \Theta_T \cup \text{Lower}(O_{7.20})$, $\Theta_F = \Theta_F \cup \text{Upper}(O_{7.27})$.

225 According to the relationship in Fig. 1,

$$\begin{aligned}\Theta_T &= \{ O_{1.0}, O_{2.0}, O_{2.1}, O_{3.0}, O_{3.1}, O_{3.2}, O_{3.3}, O_{3.4}, O_{4.0}, O_{4.2}, O_{4.3}, O_{4.9} \\ &\quad O_{4.11}, O_{5.1}, O_{5.16}, O_{6.7}, O_{6.24}, O_{7.20} \}. \\ \Theta_F &= \{ O_{7.27}, O_{8.12}, O_{8.24}, O_{9.3}, O_{9.16}, O_{10.0}, O_{10.3}, O_{10.5}, O_{10.9}, O_{10.11}, \\ &\quad O_{11.0}, O_{11.1}, O_{11.2}, O_{11.3}, O_{11.4}, O_{12.0}, O_{12.1}, O_{13.0}, O_{14.0} \}.\end{aligned}$$

Step 3: Now we consider $\Delta_f^{G_6^9}$. One can check that $H_9^1 \in O_{4.10}$, $H_9^2 \in O_{6.24} \in \Theta_T$, $H_9^1 \cup H_9^3 \in O_{8.24} \in \Theta_F$, $H_9^2 \cup H_9^3 \in O_{8.24} \in \Theta_F$, and $H_9^1 \cup H_9^2 \in O_{10.6}$. Therefore, in order to make $\chi(\Delta_f^{G_6^9}) = 1$ there are only two possible cases, $\Delta_f^{G_6^9} = \{\{H_9^2\}, \emptyset\}$ and $\Delta_f^{G_6^9} = \{\{H_9^1\}, \{H_9^2\}, \{H_9^3\}, \{H_9^1, H_9^2\}, \{H_9^3, H_9^2\}, \emptyset\}$. Suppose
230 $\Delta_f^{G_6^9} = \{\{H_9^1\}, \emptyset\}$ is true. Then there one new F-orbits and no T-orbits added . Thus, $\Theta_F = \Theta_F \cup \text{Upper}(O_{4.10})$. According to the relationship in Fig. 1,

$$\begin{aligned}\Theta_T &= \{ O_{1.0}, O_{2.0}, O_{2.1}, O_{3.0}, O_{3.1}, O_{3.2}, O_{3.3}, O_{3.4}, O_{4.0}, O_{4.2}, O_{4.3}, O_{4.9} \\ &\quad O_{4.11}, O_{5.1}, O_{5.16}, O_{6.7}, O_{6.24}, O_{7.20} \}. \\ \Theta_F &= \{ O_{4.10}, O_{5.5}, O_{5.10}, O_{6.1}, O_{6.8}, O_{6.13}, O_{6.16}, O_{6.20}, \\ &\quad O_{7.0}, O_{7.1}, O_{7.7}, O_{7.11}, O_{7.13}, O_{7.19}, O_{7.21}, O_{7.22}, O_{7.26}, O_{7.27}, O_{7.29}, \\ &\quad O_{8.0} \sim O_{8.7}, O_{8.11} \sim O_{8.16}, O_{8.19} \sim O_{8.22}, O_{8.24}, \\ &\quad O_{9.0} \sim O_{9.11}, O_{9.13} \sim O_{9.16}, \\ &\quad O_{10.0} \sim O_{10.11}, O_{11.0} \sim O_{11.4}, O_{12.0}, O_{12.1}, O_{13.0}, O_{14.0} \}.\end{aligned}$$

Step 4: Now we consider $\Delta_f^{G_6^7}$. One can check that $H_7^1 \in O_{2.1} \in \Theta_T$, $H_7^2 \in O_{6.23}$, $H_7^3 \in O_{3.1} \in \Theta_T$, $H_7^4 \in O_{3.4} \in \Theta_T$, $H_7^1 \cup H_7^3 \in O_{5.3}$, $H_7^1 \cup H_7^4 \in O_{5.15}$, $H_7^3 \cup H_7^4 \in O_{6.24} \in \Theta_T$, and for all $10 \leq k$, $O_{k,j} \subseteq \Theta_F$. Therefore, in order to
235 make $\chi(\Delta_f^{G_6^7})$ be 1, there are four possible cases,

1. $\Delta_f^{G_6^7} = \{\{H_7^1\}, \{H_7^3\}, \{H_7^4\}, \{H_7^3, H_7^4\}, \{H_7^1, H_7^3\}, \emptyset\}$;
2. $\Delta_f^{G_6^7} = \{\{H_7^1\}, \{H_7^3\}, \{H_7^4\}, \{H_7^3, H_7^4\}, \{H_7^1, H_7^4\}, \emptyset\}$;
3. $\Delta_f^{G_6^7} = \{\{H_7^1\}, \{H_7^3\}, \{H_7^4\}, \{H_7^3, H_7^4\}, \{H_7^1, H_7^3\}, \{H_7^1, H_7^4\}, \{H_7^1, H_7^3, H_7^4\}, \emptyset\}$;
4. $\Delta_f^{G_6^7} = \{\{H_7^1\}, \{H_7^2\}, \{H_7^3\}, \{H_7^4\}, \{H_7^3, H_7^4\}, \{H_7^1, H_7^3\}, \{H_7^1, H_7^4\}, \emptyset\}$.

240 Suppose the first one is true. Then there is one new T-orbits and two new
F-orbits. Thus, $\Theta_T = \Theta_T \cup \text{Upper}(O_{5.3})$ and $\Theta_F = \Theta_F \cup \text{Upper}(O_{5.15}) \cup$
 $\text{Upper}(O_{6.23})$. According to the relationship in Fig. 1,

$$\begin{aligned}\Theta_T = & \{ O_{1.0}, O_{2.0}, O_{2.1}, O_{3.0} \sim O_{3.4}, O_{4.0}, O_{4.2}, O_{4.3}, O_{4.6}, O_{4.9} \\ & O_{4.11}, O_{5.1}, O_{5.3}, O_{5.16}, O_{6.7}, O_{6.24}, O_{7.20} \}. \\ \Theta_F = & \{ O_{4.10}, O_{5.5}, O_{5.10}, O_{5.15}, O_{6.1}, O_{6.5}, O_{6.8}, O_{6.13}, O_{6.16}, O_{6.20}, O_{6.22}, O_{6.23}, \\ & O_{7.0}, O_{7.1}, O_{7.3}, O_{7.4}, O_{7.6}, O_{7.11}, O_{7.13}, O_{7.14}, O_{7.16}, O_{7.17}, O_{7.19}, O_{7.21}, O_{7.22}, \\ & O_{7.24}, O_{7.26}, O_{7.27}, O_{7.28}, O_{7.29}, \\ & O_{8.0} \sim O_{8.9}, O_{8.11} \sim O_{8.24}, O_{9.0} \sim O_{9.11}, O_{9.13} \sim O_{9.16}, \\ & O_{10.0} \sim O_{10.11}, O_{11.0} \sim O_{11.4}, O_{12.0}, O_{12.1}, O_{13.0}, O_{14.0} \}.\end{aligned}$$

Step 5: Now we consider $\Delta_f^{G_6^6}$. According to Θ_T and Θ_F , we can check that
 $H_6^1 \in O_{2.1} \in \Theta_T$, $H_6^2 \in O_{4.7}$, $H_6^3 \in O_{4.11} \in \Theta_F$, $H_6^4 \in O_{4.7}$, $H_6^1 \cup H_6^2 \in O_{6.19}$,
245 $H_6^1 \cup H_6^3 \in O_{6.24}$, $H_6^1 \cup H_6^4 \in O_{6.19}$, $H_6^2 \cup H_6^3 \in O_{8.10}$, $H_6^2 \cup H_6^4 \in O_{8.14} \in \Theta_F$,
 $H_6^3 \cup H_6^4 \in O_{8.10}$, and, again, for all $10 \leq k$, $O_{k,j} \in \Theta_F$. Therefore, in order to
make $\chi(\Delta_f^{G_6^7})$ be 1, there are three possible cases,

1. $\Delta_f^{G_6^6} = \{\{H_6^1\}, \{H_6^3\}, \{H_6^1, H_6^3\}, \emptyset\}$;
2. $\Delta_f^{G_6^6} = \{\{H_6^1\}, \{H_6^2\}, \{H_6^3\}, \{H_6^4\}, \{H_6^1, H_6^3\}, \{H_6^1, H_6^2\}, \{H_6^1, H_6^4\}, \emptyset\}$;
- 250 3. $\Delta_f^{G_6^6} = \{\{H_6^1\}, \{H_6^2\}, \{H_6^3\}, \{H_6^4\}, \{H_6^1, H_6^3\}, \{H_6^2, H_6^3\}, \{H_6^3, H_6^4\}, \emptyset\}$;

Suppose the first one is true. Then there is one new F-orbits added. Thus,
 $\Theta_F = \Theta_F \cup \text{Upper}(O_{4.7})$. According to the relationship in Fig. 1,

$$\begin{aligned}\Theta_T = & \{ O_{1.0}, O_{2.0}, O_{2.1}, O_{3.0} \sim O_{3.4}, O_{4.0}, O_{4.2}, O_{4.3}, O_{4.6}, O_{4.9} \\ & O_{4.11}, O_{5.1}, O_{5.3}, O_{5.16}, O_{6.7}, O_{6.24}, O_{7.20} \}. \\ \Theta_F = & \{ O_{4.7}, O_{4.10}, O_{5.0}, O_{5.5}, O_{5.9}, O_{5.10}, O_{5.11}, O_{5.15}, \\ & O_{6.0} \sim O_{6.6}, O_{6.8}, O_{6.9}, O_{6.13}, O_{6.15}, O_{6.16}, O_{6.19} \sim O_{6.23}, \\ & O_{7.0} \sim O_{7.19}, O_{7.21} \sim O_{7.24}, O_{7.26} \sim O_{7.29}, \\ & O_{8.0} \sim O_{8.24}, O_{9.0} \sim O_{9.16}, \\ & O_{10.0} \sim O_{10.11}, O_{11.0} \sim O_{11.4}, O_{12.0}, O_{12.1}, O_{13.0}, O_{14.0} \}.\end{aligned}$$

Table B.5: k-combinations of 1-orbits of G_6^3

k	combinations
1	$O_{1.0} \times 2, O_{3.1}, O_{3.2} \times 2, O_{3.4}$
2	$O_{2.1}, O_{4.0} \times 2, O_{4.2} \times 2, O_{4.8} \times 2, O_{4.10} \times 2,$ $O_{6.7} \times 2, O_{6.14} \times 2, O_{6.23}, O_{6.24},$
3	$O_{5.3}, O_{5.10} \times 2, O_{5.15}, O_{7.0} \times 2, O_{7.20} \times 2, O_{7.25} \times 2,$ $O_{7.27} \times 4, O_{7.29} \times 2, O_{9.12}, O_{9.13} \times 2, O_{9.14}.$

Step 6: Now we consider $\Delta_f^{G_6^3}$. The combinations of the orbits on 1-tuples under G_6^3 are shown as Table B.5. Because all the 4-combinations of H_3^i have at least 8 elements and for all $i > 7$ and j , $O_{i,j} \subseteq \Theta_F$, $\Delta_f^{G_6^3}$ can only have 1-combination, 2-combinations, or 3-combinations. According to Θ_T , $\Delta_f^{G_6^3}$ currently has 6 1-combinations, 8 2-combinations and 3 3-combinations. According to Θ_T and Θ_F , for the orbits of the 2-combinations and 3-combinations, the free orbits are $O_{4.8}$, $O_{6.14}$ and $O_{7.25}$. Furthermore, $O_{4.8} \in O_{7.25}$. Therefore, there two possible settings of $O_{4.8}$, $O_{6.14}$ and $O_{7.25}$, for $\Delta_f^{G_6^3}$ to be one. One is to set $O_{4.8}$, $O_{6.14}$ and $O_{7.25}$ as F-orbits, and the other one is to set $O_{7.25}$ and $O_{4.8}$ as T-orbits and $O_{6.14}$ as an F-orbit. Suppose the former one is true. We update Θ_T and Θ_F accordingly and obtain the followings,

$$\begin{aligned}
\Theta_T = \{ & O_{1.0}, O_{2.0}, O_{2.1}, O_{3.0} \sim O_{3.4}, O_{4.0}, O_{4.2}, O_{4.3}, O_{4.6}, O_{4.9} \\
& O_{4.11}, O_{5.1}, O_{5.3}, O_{5.16}, O_{6.7}, O_{6.24}, O_{7.20} \}. \\
\Theta_F = \{ & O_{4.7}, O_{4.8}, O_{4.10}, O_{5.0}, O_{5.4}, O_{5.5}, O_{5.9}, O_{5.10}, O_{5.11}, O_{5.13}, O_{5.15}, \\
& O_{6.0} \sim O_{6.6}, O_{6.8} \sim O_{6.10}, O_{6.13} \sim O_{6.16}, O_{6.18} \sim O_{6.23}, \\
& O_{7.0} \sim O_{7.19}, O_{7.21} \sim O_{7.29}, O_{8.0} \sim O_{8.24}, O_{9.0} \sim O_{9.16}, \\
& O_{10.0} \sim O_{10.11}, O_{11.0} \sim O_{11.4}, O_{12.0}, O_{12.1}, O_{13.0}, O_{14.0} \}.
\end{aligned}$$

Step 7: Now we are ready to consider $\chi(\Delta_f^{G_6^1})$ and $\chi(\Delta_{f_{x_1=1}})$. According to Θ_T and Θ_F , currently we have $\chi(\Delta_f^{G_6^1}) = 1$ and $\chi(\Delta_{f_{x_1=1}}) = 7$, and the only free orbits are $O_{5.4}, O_{5.6}, O_{5.12}, O_{6.10}, O_{6.12}$ and $O_{6.17}$. The size of these orbits together with their relations are shown in Fig. B.2, where the a/b implies

that orbit has totally a elements where b of them contains variable x_1 . Recall the definition of $\chi(\Delta)$ in Eq. (1), once adding an orbits $O_{i,j}$ to Θ_T , $\chi(\Delta_f^{G_6^1})$ and $\chi(\Delta_{f_{x_1=1}})$ increase $(-1)^{i+1}|O_{i,j}|$ and $(-1)^i|O_{i,j}|$, respectively. Note that
270 whatever the types these free orbits own, the values of $\chi(\Delta_f^{G_6^i})$, $2 \leq i \leq 11$ remain unchanged. Therefore, for this subcase, it suffices to show there is no setting of the types of these free orbits can satisfy both $\chi(\Delta_f^{G_6^1}) = 1$ and $\chi(\Delta_{f_{x_1=1}}) = 1$. In order to make $\chi(\Delta_f^{G_6^1})$ be 1, there are two possible cases,

- 275 1. T-orbits: $O_{6.12}, O_{6.17}, O_{5.6}, O_{5.12}$; F-orbits: $O_{6.10}, O_{5.4}$.
 2. T-orbits: $O_{6.12}, O_{6.17}, O_{5.6}, O_{5.12}, O_{6.10}, O_{5.4}$.

One can check that $\chi(\Delta_{f_{x_1=1}}) \neq 1$ in neither of the above cases. Then the checking process will back to the previous step and consider other possible cases.

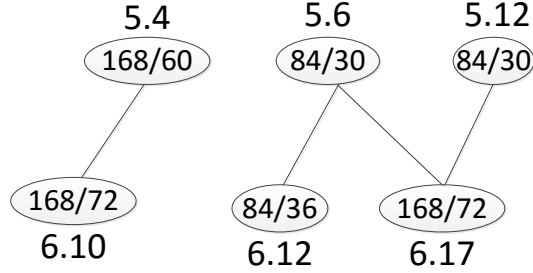


Figure B.2: **Remained free orbits**

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